27. Integrate by parts with $u = x^2$, $dv = e^{-x}dx \quad \Rightarrow \quad du = 2x\,dx$, $v = -e^{-x}$:

$$I = \int x^2e^{-x}\,dx = -x^2e^{-x} + 2\int xe^{-x}dx$$

Now integrate by parts with $u = x$, $dv = e^{-x}dx \quad \Rightarrow \quad du = dx$, $v = -e^{-x}$:

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$$

Thus, $I = -x^2e^{-x} + 2(-xe^{-x} - e^{-x} + C) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C = -e^{-x}(x^2 + 2x + 2) + C$.

28. $\int \sin^4\theta\cos^3\theta\,d\theta = \int \sin^4\theta\cos^2\theta\,\cos\theta\,d\theta = \int \sin^4\theta\,(1 - \sin^2\theta)\cos\theta\,d\theta$

$$= \int u^4(1-u^2)\,du \quad [u = \sin\theta,\ du = \cos\theta\,d\theta]$$

$$= \int (u^4 - u^6)\,du = \tfrac{1}{5}u^5 - \tfrac{1}{7}u^7 + C = \tfrac{1}{5}\sin^5\theta - \tfrac{1}{7}\sin^7\theta + C$$

29. $\dfrac{1}{t^2+6t+8} = \dfrac{1}{(t+2)(t+4)} = \dfrac{A}{t+2} + \dfrac{B}{t+4}$. Multiply both sides by $(t+2)(t+4)$ to get $1 = A(t+4) + B(t+2)$. Substituting -4 for t gives $1 = -2B \quad \Leftrightarrow \quad B = -\frac{1}{2}$. Substituting -2 for t gives $1 = 2A \quad \Leftrightarrow \quad A = \frac{1}{2}$. Thus,

$$\int \frac{dt}{t^2+6t+8} = \int\left(\frac{1/2}{t+2} - \frac{1/2}{t+4}\right)dt = \tfrac{1}{2}\ln|t+2| - \tfrac{1}{2}\ln|t+4| + C = \tfrac{1}{2}\ln\left|\frac{t+2}{t+4}\right| + C.$$

30. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}(x^2) + C$.

31. Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$ since $\cos\theta \ge 0$ for $-\pi/2 \le \theta \le \pi/2$. When $x = 0$, $3\sin\theta = 0 \quad \Rightarrow \quad \theta = 0$ and when $x = 3$, $3\sin\theta = 3 \quad \Rightarrow \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$. Thus,

$$\int x^3\sqrt{9-x^2}\,dx = \int (3\sin\theta)^3(3\cos\theta)(3\cos\theta\,d\theta) = 3^5\int \sin^3\theta\cos^2\theta\,d\theta$$

$$= 3^5\int \sin^2\theta\cos^2\theta(\sin\theta\,d\theta) = 3^5\int (1-\cos^2\theta)\cos^2\theta(\sin\theta\,d\theta) = I.$$

Now let $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$ and when $\theta = \frac{\pi}{2}$, $u = 0$. Substitution gives us

$$I = 3^5\int (1-u^2)u^2(-du) = 3^5\int (u^2 - u^4)du = 3^5\left[\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right]$$

Another method: Let $u = 9 - x^2$. Then $du = -2x\,dx$ and $x^2 = 9 - u$, so

$$\int x^3\sqrt{9-x^2}\,dx = \int x^2\sqrt{9-x^2}\,(x\,dx) = \int (9-u)\sqrt{u}\left(-\tfrac{1}{2}\,du\right) = \frac{1}{2}\int\left(9u^{1/2} - u^{3/2}\right)du$$

$$= \tfrac{1}{2}\left[6u^{3/2} - \tfrac{2}{5}u^{5/2}\right]$$

32. Let $u = \tan^{-1}x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{1+x^2}\,dx$, $v = x$:

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{1+x^2}\,dx = x\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2) + C.$$

33. Let $u = 1 + \sec\theta$. Then $du = \sec\theta\,\tan\theta\,d\theta$, so

$$\int\frac{\sec\theta\,\tan\theta}{1+\sec\theta}\,d\theta = \int\frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C$$