28. $\int \sin^4\theta\cos^3\theta\,d\theta = \int \sin^4\theta\cos^2\theta\cos\theta\,d\theta = \int \sin^4\theta\left(1-\sin^2\theta\right)\cos\theta\,d\theta$

$= \int u^4\left(1-u^2\right)du \quad [u=\sin\theta,\ du=\cos\theta\,d\theta]$

$= \int \left(u^4-u^6\right)du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + C = \frac{1}{5}\sin^5\theta - \frac{1}{7}\sin^7\theta + C$

29. $\dfrac{1}{t^2+6t+8} = \dfrac{1}{(t+2)(t+4)} = \dfrac{A}{t+2} + \dfrac{B}{t+4}$. Multiply both sides by $(t+2)(t+4)$ to get $1 = A(t+4)+B(t+2)$. Substituting -4 for t gives $1=-2B \quad\Leftrightarrow\quad B=-\frac{1}{2}$. Substituting -2 for t gives $1=2A \quad\Leftrightarrow\quad A=\frac{1}{2}$. Thus,

$$\int \frac{dt}{t^2+6t+8} = \int\left(\frac{1/2}{t+2}-\frac{1/2}{t+4}\right)dt = \tfrac{1}{2}\ln|t+2| - \tfrac{1}{2}\ln|t+4| + C = \tfrac{1}{2}\ln\left|\frac{t+2}{t+4}\right| + C.$$

30. Let $u=x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int\frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\sin^{-1}\left(x^2\right)+C.$

31. Let $x=3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$ since $\cos\theta \ge 0$ for $-\pi/2\le\theta\le\pi/2$. When $x=0$, $3\sin\theta=0 \;\Rightarrow\; \theta=0$ and when $x=3$, $3\sin\theta=3 \;\Rightarrow\; \sin\theta=1 \;\Rightarrow\; \theta=\frac{\pi}{2}$. Thus,

$$\int_0^3 x^3\sqrt{9-x^2}\,dx = \int_0^{\pi/2}(3\sin\theta)^3(3\cos\theta)(3\cos\theta\,d\theta) = 3^5\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$= 3^5\int_0^{\pi/2}\sin^2\theta\cos^2\theta(\sin\theta\,d\theta) = 3^5\int_0^{\pi/2}\left(1-\cos^2\theta\right)\cos^2\theta(\sin\theta\,d\theta) = I.$$

Now let $u=\cos\theta$ so that $du=-\sin\theta\,d\theta$. When $\theta=0$, $u=1$ and when $\theta=\frac{\pi}{2}$, $u=0$. Substitution gives us

$$I = 3^5\int_1^0\left(1-u^2\right)u^2(-du) = 3^5\int_0^1\left(u^2-u^4\right)du = 3^5\left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]_0^1$$

$$= 3^5\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = 3^5\left(\tfrac{2}{15}\right) = \tfrac{162}{5} = 32.4.$$

Another method: Let $u=9-x^2$. Then $du=-2x\,dx$ and $x^2=9-u$, so

$$\int_0^3 x^3\sqrt{9-x^2}\,dx = \int_0^3 x^2\sqrt{9-x^2}\,(x\,dx) = \int_9^0(9-u)\sqrt{u}\left(-\tfrac{1}{2}\,du\right) = \frac{1}{2}\int_0^9\left(9u^{1/2}-u^{3/2}\right)du$$

$$= \tfrac{1}{2}\left[6u^{3/2}-\tfrac{2}{5}u^{5/2}\right]_0^9 = \tfrac{1}{2}\left[\left(6\cdot 27-\tfrac{2}{5}\cdot 243\right)-0\right] = \tfrac{1}{2}\left(\tfrac{324}{5}\right) = \tfrac{162}{5} = 32.4$$

32. Let $u=\tan^{-1}x$, $dv=dx \;\Rightarrow\; du=\dfrac{1}{1+x^2}\,dx$, $v=x$:

$$\int\tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{1+x^2}\,dx = x\tan^{-1}x - \tfrac{1}{2}\ln\left(1+x^2\right)+C.$$

33. Let $u=1+\sec\theta$. Then $du=\sec\theta\,\tan\theta\,d\theta$, so

$$\int\frac{\sec\theta\,\tan\theta}{1+\sec\theta}\,d\theta = \int\frac{1}{u}\,du = \ln|u|+C = \ln|1+\sec\theta|+C$$