22. Let $u = \tan^{-1} x$, $dv = x\,dx \quad \Rightarrow \quad du = dx/\left(1+x^2\right)$, $v = \frac{1}{2}x^2$.

Then $\int x \tan^{-1} x\,dx = \frac{1}{2}x^2 \tan^{-1} x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{\left(1+x^2\right)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1} x + C_1$. So

$\int x \tan^{-1} x\,dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x + C_1\right) = \frac{1}{2}\left(x^2 \tan^{-1} x + \tan^{-1} x - x\right) + C$.

41. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \quad \Rightarrow \quad du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2 e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \quad \Rightarrow \quad dU = dw$, $V = -e^{-w}$. Then

$$\begin{aligned} s(t) &= -t^2 e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2 e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right) \\ &= -t^2 e^{-t} + 2\left(-te^{-t} - e^{-t} + 1\right) = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2 \\ &= 2 - e^{-t}\left(t^2 + 2t + 2\right) \text{ meters} \end{aligned}$$

42. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$\begin{aligned} H &= \int_0^{60} \left[-gt - v_e \ln\left(\frac{m-rt}{m}\right)\right] dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60} \ln(m-rt)\,dt - \int_0^{60} \ln m\,dt\right] \\ &= -g(1800) + v_e(\ln m)(60) - v_e \int_0^{60} \ln(m-rt)\,dt \end{aligned}$$

Let $u = \ln(m-rt)$, $dv = dt \quad \Rightarrow \quad du = \dfrac{1}{m-rt}(-r)\,dt$, $v = t$. Then

$$\begin{aligned} \int_0^{60} \ln(m-rt)\,dt &= \left[t\ln(m-rt)\right]_0^{60} + \int_0^{60} \frac{rt}{m-rt}\,dt = 60\ln(m-60r) + \int_0^{60}\left(-1 + \frac{m}{m-rt}\right)dt \\ &= 60\ln(m-60r) + \left[-t - \frac{m}{r}\ln(m-rt)\right]_0^{60} \\ &= 60\ln(m-60r) - 60 - \frac{m}{r}\ln(m-60r) + \frac{m}{r}\ln m \end{aligned}$$

So $H = -1800g + 60v_e \ln m - 60v_e \ln(m-60r) + 60v_e + \dfrac{m}{r}v_e \ln(m-60r) - \dfrac{m}{r}v_e \ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.