7.3 Exercises

1–8 ■ Solve the differential equation.

1. $\dfrac{dy}{dx} = y^2$

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3}$

3. $yy' = x$

4. $y' = xy$

5. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}}$

6. $y' = \dfrac{xy}{2\ln y}$

7. $\dfrac{du}{dt} = 2 + 2u + t + tu$

8. $\dfrac{dz}{dt} + e^{t+z} = 0$

9–14 ■ Find the solution of the differential equation that satisfies the given initial condition.

9. $\dfrac{dy}{dx} = y^2 + 1, \quad y(1) = 0$

10. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}, \quad y(0) = 1$

11. $xe^{-t}\dfrac{dx}{dt} = t, \quad x(0) = 1$

12. $x + 2y\sqrt{x^2+1}\,\dfrac{dy}{dx} = 0, \quad y(0) = 1$

13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

14. $\dfrac{dy}{dt} = te^y, \quad y(1) = 0$

15. Find an equation of the curve that satisfies $dy/dx = 4x^3y$ and whose y-intercept is 7.

16. Find an equation of the curve that passes through the point $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

29. In Exercise 13 in Section 7.1 we formulated a model for learning in the form of the differential equation

$$\frac{dP}{dt} = k(M - P)$$

where $P(t)$ measures the performance of someone learning a skill after a training time t, M is the maximum level of performance, and k is a positive constant. Solve this differential equation to find an expression for $P(t)$. What is the limit of this expression?

30. In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: $A + B \longrightarrow C$. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B]$$

(See Example 4 in Section 3.3.) Thus, if the initial concentrations are $[A] = a$ moles/L and $[B] = b$ moles/L and we write $x = [C]$, then we have

$$\frac{dx}{dt} = k(a - x)(b - x)$$

CAS (a) Assuming that $a \neq b$, find x as a function of t. Use the fact that the initial concentration of C is 0.
(b) Find $x(t)$ assuming that $a = b$. How does this expression for $x(t)$ simplify if it is known that $[C] = a/2$ after 20 seconds?

31. In contrast to the situation of Exercise 30, experiments show that the reaction $H_2 + Br_2 \longrightarrow 2HBr$ satisfies the rate law

$$\frac{d[HBr]}{dt} = k[H_2][Br_2]^{1/2}$$

and so for this reaction the differential equation becomes

$$\frac{dx}{dt} = k(a - x)(b - x)^{1/2}$$

where $x = [HBr]$ and a and b are the initial concentrations of hydrogen and bromine.
(a) Find x as a function of t in the case where $a = b$. Use the fact that $x(0) = 0$.
(b) If $a > b$, find t as a function of x. [*Hint:* In performing the integration, make the substitution $u = \sqrt{b - x}$.]

32. A sphere with radius 1 m has temperature 15 °C. It lies inside a concentric sphere with radius 2 m and temperature 25 °C. The temperature $T(r)$ at a distance r from the common center of the spheres satisfies the differential equation

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0$$

If we let $S = dT/dr$, then S satisfies a first-order differential equation. Solve it to find an expression for the temperature $T(r)$ between the spheres.

33. A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances and removed from the bloodstream at a rate that is proportional to the concentration at that time. Thus, a model for the concentration $C = C(t)$ of the glucose solution in the bloodstream is

$$\frac{dC}{dt} = r - kC$$

where k is a positive constant.
(a) Suppose that the concentration at time $t = 0$ is C_0. Determine the concentration at any time t by solving the differential equation.
(b) Assuming that $C_0 < r/k$, find $\lim_{t\to\infty} C(t)$ and interpret your answer.

34. A certain small country has \$10 billion in paper currency in circulation, and each day \$50 million comes into the country's banks. The government decides to introduce new currency by having the banks replace old bills with new ones whenever old currency comes into the banks. Let $x = x(t)$ denote the amount of new currency in circulation at time t, with $x(0) = 0$.
(a) Formulate a mathematical model in the form of an initial-value problem that represents the "flow" of the new currency into circulation.
(b) Solve the initial-value problem found in part (a).
(c) How long will it take for the new bills to account for 90% of the currency in circulation?

35. A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes?