7.3 Separable Equations

1. $\frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{y^2} = dx \; [y \neq 0] \Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + C \Rightarrow -y = \frac{1}{x+C} \Rightarrow$
$y = \frac{-1}{x+C}$, and $y = 0$ is also a solution.

2. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3} \Rightarrow 4y^3\,dy = e^{2x}\,dx \Rightarrow \int 4y^3\,dy = \int e^{2x}\,dx \Rightarrow y^4 = \frac{1}{2}e^{2x} + C \Rightarrow$
$y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

3. $yy' = x \Rightarrow y\frac{dy}{dx} = x \Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1 \Rightarrow y^2 = x^2 + 2C_1 \Rightarrow$
$x^2 - y^2 = C$ (where $C = -2C_1$). This represents a family of hyperbolas.

4. $y' = xy \Rightarrow \int \frac{dy}{y} = \int x\,dx \;\; [y \neq 0] \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow |y| = e^C e^{x^2/2} \Rightarrow y = Ke^{x^2/2}$,
where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

5. $\frac{dy}{dt} = \frac{te^t}{y\sqrt{1+y^2}} \Rightarrow y\sqrt{1+y^2}\,dy = te^t\,dt \Rightarrow \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \Rightarrow$
$\frac{1}{3}(1+y^2)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$
$1 + y^2 = [3(te^t - e^t + C)]^{2/3} \Rightarrow y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

6. $y' = \frac{xy}{2\ln y} \Rightarrow \frac{2\ln y}{y}\,dy = x\,dx \Rightarrow \int \frac{2\ln y}{y}\,dy = \int x\,dx \Rightarrow (\ln y)^2 = \frac{x^2}{2} + C \Rightarrow$
$\ln y = \pm\sqrt{x^2/2 + C} \Rightarrow y = e^{\pm\sqrt{x^2/2+C}}$

7. $\frac{du}{dt} = 2 + 2u + t + tu \Rightarrow \frac{du}{dt} = (1+u)(2+t) \Rightarrow \int \frac{du}{1+u} = \int (2+t)dt \;\; [u \neq -1] \Rightarrow$
$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \Rightarrow |1+u| = e^{t^2/2 + 2t + C} = Ke^{t^2/2+2t}$, where $K = e^C \Rightarrow$
$1 + u = \pm Ke^{t^2/2+2t} \Rightarrow u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so
$u = -1 + Ae^{t^2/2+2t}$, where A is an arbitrary constant.

8. $\frac{dz}{dt} + e^{t+z} = 0 \Rightarrow \frac{dz}{dt} = -e^t e^z \Rightarrow \int e^{-z}\,dz = -\int e^t\,dt \Rightarrow -e^{-z} = -e^t + C \Rightarrow e^{-z} = e^t - C$
$\Rightarrow \frac{1}{e^z} = e^t - C \Rightarrow e^z = \frac{1}{e^t - C} \Rightarrow z = \ln\left(\frac{1}{e^t - C}\right) \Rightarrow z = -\ln(e^t - C)$

9. $\frac{dy}{dx} = y^2 + 1$, $y(1) = 0$. $\int \frac{dy}{y^2+1} = \int dx \Rightarrow \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so
$1 + C = \tan^{-1} 0 = 0 \Rightarrow C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x-1)$.

10. $\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \Rightarrow \frac{1+y^2}{y}dy = \cos x\,dx \Rightarrow$
$\int\left(\frac{1}{y} + y\right)dy = \int \cos x\,dx \Rightarrow \ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow$
$C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.