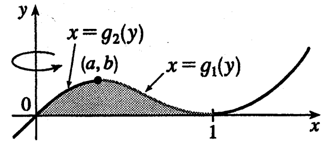
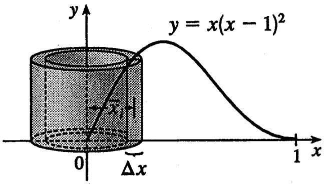
43. 

If we were to use the "washer method," we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the figure above. This step would be difficult because it involves the cubic formula. Finally we would find the volume using

$$V = \pi \int_0^b \left\{ [g_1(y)]^2 - [g_2(y)]^2 \right\} dy.$$



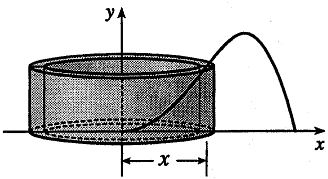
Instead, we use cylindrical shells. As in Example 9, we rotate an approximating rectangle with width Δx about the y-axis, to get a cylindrical shell whose average radius is $\overline{x}_i$ and whose volume is $2\pi\overline{x}_i\left[\overline{x}_i(\overline{x}_i-1)^2\right]\Delta x$.

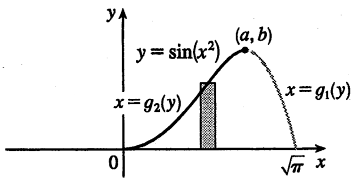
So the total volume is

$$V=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi\,\overline{x}_i\left[\overline{x}_i(\overline{x}_i-1)^2\right]\Delta x=\int_0^1 2\pi x\left[x(x-1)^2\right]dx=2\pi\int_0^1\left(x^4-2x^3+x^2\right)dx$$

$$=2\pi\left[\frac{x^5}{5}-2\frac{x^4}{4}+\frac{x^3}{3}\right]_0^1=2\pi\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right)=2\pi\left(\frac{1}{30}\right)=\frac{\pi}{15}.$$

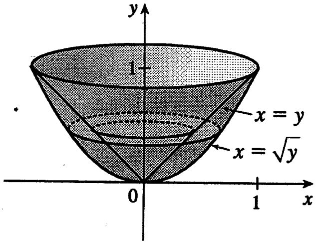
44.





A typical cylindrical shell has circumference $2\pi x$ and height $\sin\left(x^2\right)$. $V=\int_0^{\sqrt{\pi}}2\pi x\sin\left(x^2\right)dx$. Let $u=x^2$. Then $du=2x\,dx$, so $V=\pi\int_0^{\pi}\sin u\,du=\pi[-\cos u]_0^{\pi}=\pi[1-(-1)]=2\pi$. For slicing, we would first have to locate the local maximum point (a,b) of $y=\sin\left(x^2\right)$ using the methods of Chapter 4. Then we would have to solve the equation $y=\sin\left(x^2\right)$ for x in terms of y to obtain the functions $x=g_1(y)$ and $x=g_2(y)$ shown in the second figure. Finally we would find the volume using $V=\pi\int_0^b\left\{[g_1(y)]^2-[g_2(y)]^2\right\}dy$. Using shells is definitely preferable to slicing.

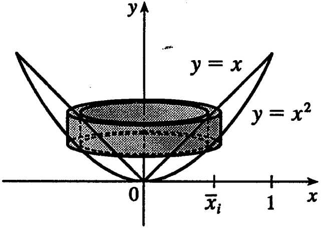
46.



The first figure shows a cross-section perpendicular to the y-axis. It is a washer with inner radius y and outer radius $\sqrt{y}$, so the volume by slicing is

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 \left[\left(\sqrt{y}\right)^2 - y^2\right] dy$$

$$= \pi \int_0^1 (y - y^2)\,dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{\pi}{6}$$

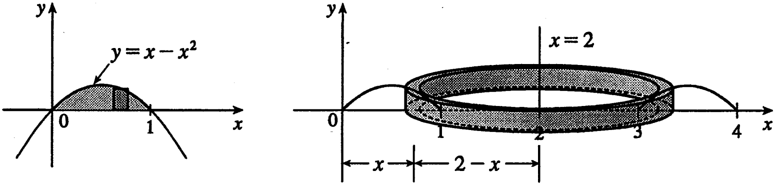


Next, we use cylindrical shells to find the volume. The height of a shell is $\overline{x}_i - \overline{x}_i^2$, so its volume is $2\pi\,\overline{x}_i\left(\overline{x}_i - \overline{x}_i^2\right)\Delta x$. Thus,

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\,\overline{x}_i\left(\overline{x}_i - \overline{x}_i^2\right)\Delta x = \int_0^1 2\pi x\left(x - x^2\right) dx$$

$$= 2\pi \int_0^1 \left(x^2 - x^3\right) dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{12}\right) = \frac{\pi}{6}$$

47.



$$V = \int_0^1 (\text{circumference})\,(\text{height})\,(\text{thickness}) = \int_0^1 \left[2\pi(2 - x)\right]\left(x - x^2\right) dx$$

$$= 2\pi \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx = 2\pi\left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^1 = 2\pi\left(\tfrac{1}{4}\right) = \tfrac{\pi}{2}$$

See the solution for Exercise 43 as to why the method of cylindrical shells is preferable to slicing.